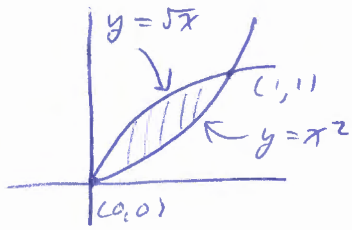
Math 31 - Summer 2013 **Second Exam** Name_______________

**Instructions:** You must show all work to receive full credit. All problems are worth 8points.

1. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$.




$\sqrt{x} = x^2$ So the graphs intersect when
$x = x^4$ $x = 0, 1$ or at $(0,0)$ and $(1,1)$
$0 = x(x^3 - 1)$

$$\text{Area } A = \int_0^1 (\sqrt{x} - x^2)\,dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right|_0^1$$

$$= \boxed{\frac{1}{3}}$$

2. Find the average value of the function $f(x) = \dfrac{1}{2\sqrt{x}}$ on the interval [1, 9]

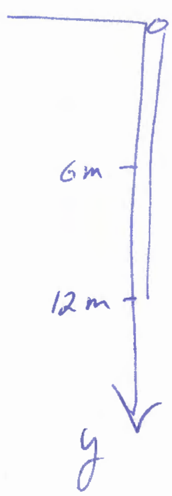
$$f_{ave} = \frac{1}{9-1}\int_1^9 \frac{1}{2\sqrt{x}}\,dx = \frac{1}{8}\sqrt{x}\Big|_1^9 = \frac{1}{8}(3-1) = \frac{2}{8} = \boxed{\frac{1}{4}}$$

3. A 12-meter chain hangs from the top of a building. If the chain has a mass of 72 kg, how much work is required to pull up 6 meters of chain? (The acceleration due to gravity is 9.8 m/s$^2$.) (Hint: Drawing a picture may help.)

mass density: $\frac{72\text{ kg}}{12\text{ m}} = 6\text{ kg/m}$

weight density $= 6\frac{\text{kg}}{\text{m}} \cdot 9.8\frac{\text{m}}{\text{s}^2} = 58.8\frac{\text{N}}{\text{m}}$

Note: This is really two problems. The top half of the chain moves a variable distance depending on position, but the bottom half moves a fixed distance of 6m.

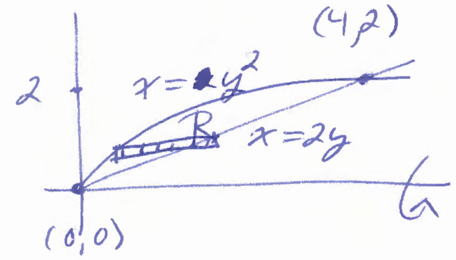



$$\text{Work, } W = \int_0^6 58.8\,y\,dy + (58.8)(6)(6)$$

$$= 29.4y^2\Big|_0^6 + (58.8)(36) = \boxed{3175.2\text{ J}}$$

4. Use shells to find the volume of the solid obtained by rotating the region bounded by the curves $y=\frac{1}{2}x$, $y=\sqrt{x}$ about the $x$ - axis.

curves intersect at $(0,0)$ and $(4,2)$



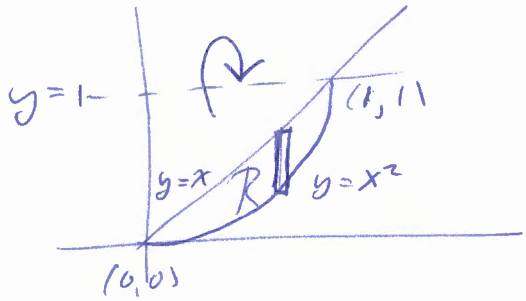

$$V = \int_0^2 2\pi y\,(2y - y^2)\,dy$$

$$= 2\pi \int_0^2 (2y^2 - y^3)\,dy$$

$$= 2\pi \left[\tfrac{2}{3}y^3 - \tfrac{1}{4}y^4\right|_0^2 = 2\pi\left[\tfrac{16}{3} - 4\right]$$

$$= 2\pi\left[\tfrac{16-12}{3}\right] = \boxed{\tfrac{8}{3}\pi}$$

5. Use washers to find the volume of the solid obtained by rotating the region bounded by the curves $y=x$, $y=x^2$ about the line $y=1$.




$$V = \int_0^1 \pi\left[(1-x^2)^2 - (1-x)^2\right]dx$$

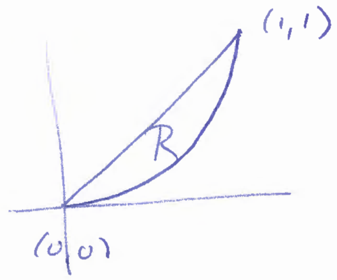
$$= \pi \int_0^1 \left[(1-2x^2+x^4) - (1-2x+x^2)\right]dx$$

$$= \pi \int_0^1 (2x - 3x^2 + x^4)\,dx$$

$$= \pi\left[x^2 - x^3 + \tfrac{1}{5}x^5\right|_0^1$$

$$= \pi\left[1 - 1 + \tfrac{1}{5}\right] = \boxed{\tfrac{1}{5}\pi}$$

6. Find the centroid (center of mass) of the region bounded by the curves $y=x$, $y=x^2$.




$$A = \int_0^1 (x - x^2)\,dx = \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{1}{6}$$

$$\int_0^1 x(x - x^2)\,dx = \int_0^1 (x^2 - x^3)\,dx =$$

$$\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\Big|_0^1 = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12}$$

$$\therefore\ \bar{x} = \frac{1/12}{1/6} = \boxed{\tfrac{1}{2}}$$

$$\int_0^1 \tfrac{1}{2}\left[(x)^2 - (x^2)^2\right]dx = \tfrac{1}{2}\int_0^1 (x^2 - x^4)\,dx = \tfrac{1}{2}\left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right|_0^1 = \tfrac{1}{15}$$

$$\therefore\ \bar{y} = \frac{1/15}{1/6} = \boxed{\tfrac{2}{5}}$$

7. Set up and simplify, but do not evaluate, an integral for the length of the curve $y=\sqrt{x}$ from (0, 0) to (4, 2).

$$y = \sqrt{x}$$
$$y' = \frac{1}{2\sqrt{x}}$$
$$L = \int_0^4 \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx = \int_0^4 \sqrt{1+\frac{1}{4x}}\,dx = \int_0^4 \sqrt{\frac{4x+1}{4x}}\,dx$$
$$L = \int_0^4 \frac{1}{2}\sqrt{\frac{4x+1}{x}}\,dx = \boxed{\frac{1}{2}\int_0^4 \sqrt{\frac{4x+1}{x}}\,dx}$$

8. Find the area of the surface obtained by rotating $y=\sqrt{x}$ about the $x$–axis from $x=0$ to $x=2$. (Unlike problem #7 above, you **are** expected to evaluate the resulting integral.)

$$y = \sqrt{x}, \quad y' = \frac{1}{2\sqrt{x}}$$
$$S = \int_0^2 2\pi\sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx \quad \text{(See Problem 7)}$$
$$= \int_0^2 \pi\sqrt{4x+1}\,dx \qquad \text{Let } u = 4x+1, \quad du = 4x\,dx, \quad \tfrac{1}{4}du = x\,dx$$

| x | u |
|---|---|
| 0 | 1 |
| 2 | 9 |

$$= \frac{\pi}{4}\int_1^9 \sqrt{u}\,du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\Big|_1^9\right.$$
$$= \frac{\pi}{6}\left[27-1\right] = \frac{26}{6}\pi = \boxed{\frac{13}{3}\pi}$$

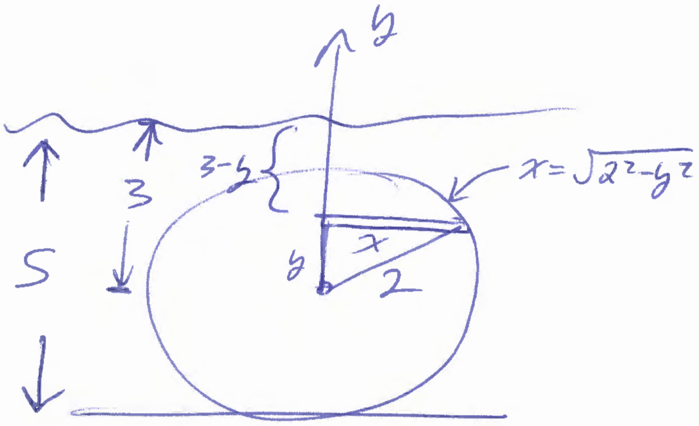
9. If 2.5 J of work is required to stretch a spring from its natural length of 20 cm to a length of 30 cm, how much work is required to stretch it from 25 cm to 30 cm?

$$2.5 = \int_0^{0.1} kx\,dx = \frac{1}{2}kx^2\Big|_0^{.1} = 0.005k, \quad \text{so } k = 500$$
$$W = \int_{0.05}^{0.1} 500x\,dx = 250x^2\Big|_{0.05}^{0.1} = \boxed{1.875 \text{ J}}$$

10. Find the hydrostatic force on one end of a cylindrical drum with radius 2 meters that is submerged in 5 meters of water. (The density of water is 1000 kg/m$^3$, and the acceleration due to the Earth's gravitational field is 9.8 m/s$^2$)

$$A_i = 2x_i \Delta y$$
$$= 2\sqrt{4-y_i^2}\,\Delta y$$

$$F_i = \rho g d_i A_i$$
$$= (1000)(9.8)(3-y_i)(2\sqrt{4-y_i^2}\,\Delta y)$$




$$F = 19{,}600 \int_{-2}^{2} (3-y)\sqrt{4-y^2}\,dy$$

$$= 19{,}600\left[\int_{-2}^{2} 3\sqrt{4-y^2}\,dy - \int_{-2}^{2} y\sqrt{4-y^2}\,dy\right]$$

Now, $\int_{-2}^{2} \sqrt{4-y^2}\,dy$ is equal to half the area of a circle of radius 2, so

$$\int_{-2}^{2} \sqrt{4-y^2}\,dy = \tfrac{1}{2}\pi 2^2 = 2\pi$$

And, $\int_{-2}^{2} y\sqrt{4-y^2}\,dy = 0$ because we're integrating an odd function over a symmetric interval about 0

So $F = (19{,}600)(3)(2\pi) = \boxed{117{,}600\pi \text{ N}}$